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On bi-Hamiltonian formulation of the perturbed Kepler problem

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Abstract

The perturbed Kepler problem is shown to be a bi-Hamiltonian system in spite of the fact that the graph of the Hamilton function is not a hypersurface of translation, which is against a necessary condition for the existence of the bi-Hamiltonian structure according to the Fernandes theorem. In fact, both the initial and perturbed Kepler systems are isochronous systems and, therefore, the Fernandes theorem cannot be applied to them.

1 Introduction

Over the last few years Magri's approach [9] to integrability through bi-Hamiltonian structures had become one of the powerful methods of integrability of evolution equations applicable in studying both finite and infinite dimensional dynamical systems.

The global, topological obstructions to the existence of a bi-Hamiltonian structure for a general completely integrable Hamiltonian system are discussed in [3, 8, 13, 16]. Some counterexamples were given to show that an existence of a bi-Hamiltonian structure is not always satisfied around a Liouville torus for a given Arnold-Liouville system. For instance, Fernandes [8] and Olver [13] announced that the perturbed Kepler problem is a completely integrable system without a bi-Hamiltonian formulation with respect to non-degenerate compatible Poisson structures in contrast with the initial Kepler problem.

Below we explicitly present a few non-degenerate bi-Hamiltonian formulations of the perturbed Kepler problem using the Bogoyavlenskij construction of a continuum of compatible Poisson structures for the isochronous Hamiltonian systems [3].

A bi-Hamiltonian vector field is one which allows two Hamiltonian formulations

$$X = PdH = P'dK\,. \tag{1.1}$$

Here P and P' are the two compatible Poisson bivectors with vanishing Schouten brackets

$$[P,P] = [P,P'] = [P',P'] = 0\,. \tag{1.2}$$

In generic cases bivectors P and P' could be degenerate and Hamiltonians H and K could be functionally dependent. However, under an additional assumption one can construct a complete sequence of functionally independent first integrals of X [3, 9, 10].

The aim of this note is to present a bi-Hamiltonian formulation of the perturbed Kepler vector field X defined by the Hamilton function

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2} - \frac{1}{r} + \frac{\epsilon}{2r^2}\,, \qquad r = \sqrt{x^2+y^2+z^2}\,, \tag{1.3}$$

and canonical Poisson bivector

$$P = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}. \tag{1.4}$$

The r^{-2} correction can be seen as due to an asymmetric mass distribution of the attracting body (e.g., the gravitational attraction of the earth on a nearby orbiting artificial satellite) or to non-Newtonian perturbation from the theory of general relativity (e.g., the motion of a particle in the Schwarzschild spherically symmetric solution of the Einstein equations), see [17].

2 Bi-Hamiltonian formulation of the perturbed Kepler problem

Following to [3, 8] we start with a discussion of the bi-Hamiltonian vector fields in terms of the action-angle variables.

According to the classical Arnold-Liouville theory in the action-angle variables J_k and ω_k a given Hamiltonian vector field $X = PdH$ has the simple form

$$X: \qquad \dot{J}_k = 0\,, \qquad \dot{\omega}_k = \frac{\partial H}{\partial J_k}\,, \qquad k = 1, \ldots, n. \tag{2.5}$$

Here $H = H(J_1, \ldots, J_n)$ is a Hamilton function and the Poisson bivector P is the canonical one

$$P = \sum_{k=1}^{n} \frac{\partial}{\partial J_k} \wedge \frac{\partial}{\partial \omega_k}\,. \tag{2.6}$$

The problem of the existence of the action-angle variables in the neighborhood of an orbit, of a level set or globally is discussed in [6], see also the recent review [12] and references within. We will look for a bi-Hamiltonian formulation only in the domain of definition of the action-angle variables.

The vector field X is called non-degenerate or anisochronous if the Kolmogorov condition for the Hessian matrix

$$\det \left| \frac{\partial^2 H(J_1, \ldots, J_n)}{\partial J_i \partial J_k} \right| \neq 0 \tag{2.7}$$

is met almost everywhere in the given action-angle coordinates. This condition implies that the dense subsets of the invariant n-dimensional tori of X are closures of trajectories.

In [3] Bogoyavlenskij proposed a complete classification of the invariant Poisson structures for non-degenerate and degenerate Hamiltonian systems, see Theorem 1 and Theorem 8, respectively.

Let us consider one trivial example of the generic Bogoyavlenskij construction for the degenerate or isochronous Hamiltonian system. If in the domain of definition of the action-angle variables we have some nonzero derivative

$$a = \frac{\partial H}{\partial J_m} \neq 0\,,$$

we can make the following canonical transformation

$$\begin{aligned} &\tilde{J}_k = J_k\,, \qquad \tilde{\omega}_k = \omega_k - \frac{\partial H}{\partial J_k}\, a^{-1}\, \omega_m\,, \qquad k \neq m \\ &\tilde{J}_m = H\,, \qquad \tilde{\omega}_m = a^{-1}\omega_m\,. \end{aligned} \tag{2.8}$$

This canonical transformation does not add new singularities to the initial action-angle variables and reduces the Hamiltonian to the simplest form

$$H = \tilde{J}_m.$$

It allows us to construct bi-Hamiltonian formulation of the initial vector field X with two functionally dependent Hamiltonians

$$H = \tilde{J}_m \qquad \text{and} \qquad K = g(\tilde{J}_m)\,, \tag{2.9}$$

but with the non-degenerate second Poisson bivector

$$P' = \sum_{k\neq m}^{n} \beta_k(\tilde{J}_k)\, \frac{\partial}{\partial \tilde{J}_k} \wedge \frac{\partial}{\partial \omega_k} + \left(\frac{dg}{d\tilde{J}_m}\right)^{-1} \frac{\partial}{\partial \tilde{J}_m} \wedge \frac{\partial}{\partial \tilde{\omega}_m}\,, \tag{2.10}$$

where $\beta_k(\tilde{J}_k)$ are arbitrary nonzero functions and $g(\tilde{J}_m)$ is such that $g' \neq 0$. In this case the eigenvalues of the corresponding recursion operator $N = P'P^{-1}$ are integrals of motion only, see examples of this type bi-Hamiltonian formulations of the Kepler problem in [3, 11, 16].

In fact the Bogoyavlenskij theorems are much more powerful, however, in the following discussion, we need only this particular case.

2.1 Action-angle variables for the perturbed Kepler problem

Let us introduce the spherical coordinates r, θ and ϕ

$$r = \sqrt{x^2 + y^2 + z^2}\,, \qquad \theta = \arctan\left(\frac{y}{x}\right)\,, \qquad \phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

which are the radius, longitude and azimuth, respectively. In order to describe the corresponding momenta we can use the so-called Mathieu generating function

$$F = p_x r \sin\phi \cos\theta + p_y r \sin\phi \sin\theta + p_z r \cos\phi\,,$$

so that

$$p_r = \frac{\partial F}{\partial r}\,, \qquad p_\theta = \frac{\partial F}{\partial \theta}\,, \qquad p_\phi = \frac{\partial F}{\partial \phi}\,.$$

In the spherical variables Hamiltonian H (1.3) takes the form

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) - \frac{1}{r} + \frac{\epsilon}{2r^2}\,,$$

such that the Hamilton-Jacobi equation $H = h$ has an additive separable solution

$$S = S_r(r) + S_\theta(\theta) + S_\phi(\phi)$$

that allows us to introduce the action-angle variables. Let us begin with the definition of two additional commuting integrals of motion

$$\ell^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\,, \qquad m = p_\phi$$

which are the total angular momentum and the component of the angular momentum along the polar axis. Then for $H = h < 0$ we can calculate the action variables

$$\begin{aligned}
J_\phi &= \frac{1}{2\pi}\oint p_\phi d\phi = m\,, \\
J_\theta &= \frac{1}{2\pi}\oint p_\theta d\theta = \frac{1}{2\pi}\oint \sqrt{\ell^2 - \frac{m^2}{sin^2\theta}}\, d\theta = \ell - m\,, \\
J_r &= \frac{1}{2\pi}\oint p_r dr = \frac{1}{2\pi}\oint \sqrt{2h + \frac{2}{r} - \frac{\ell^2 + \epsilon}{r^2}}\, dr = \frac{1}{\sqrt{-2h}} - \sqrt{\ell^2 + \epsilon}
\end{aligned} \tag{2.11}$$

using the standard integration method [2]. Then the corresponding angle variables can be obtained from the Jacobi equations

$$\omega_r = \frac{\partial S}{\partial J_r} = -rp_r\sqrt{-2h} + \arccos\frac{1+2rh}{\sqrt{1+2h(\ell^2+\epsilon)}}\,,$$

$$\omega_\theta = \frac{\partial S}{\partial J_\theta} = \frac{\ell}{\sqrt{\ell^2+\epsilon}}\left(\arccos\frac{1-\frac{\ell^2+\epsilon}{r}}{\sqrt{1+2h(\ell^2+\epsilon)}} - \omega_r\right) + \arcsin\frac{\ell\cos\theta}{\sqrt{\ell^2-m^2}}\,, \tag{2.12}$$

$$\omega_\phi = \frac{\partial S}{\partial J_\phi} = \omega_\theta + \phi + \arcsin\frac{m\cot\theta}{\sqrt{\ell^2-m^2}}\,.$$

The Hamiltonian H (1.3) in these action-angle variables takes the form

$$H = -\frac{1}{2\left(J_r + \sqrt{(J_\theta+J_\phi)^2+\epsilon}\right)^2}\,. \tag{2.13}$$

It is easy to prove that the graph of H (2.13) is not a hypersurface of translation in the action variables (2.11) [8] in contrast with the initial Kepler problem at $\epsilon = 0$.

The perturbed Kepler problem at $\epsilon \neq 0$ and the unperturbed Kepler problem at $\epsilon = 0$ are degenerate or isochronous systems

$$\det\left|\frac{\partial^2 H(J_1,\ldots,J_n)}{\partial J_i \partial J_k}\right| = 0$$

with well-defined derivative for $H = h < 0$

$$a = \frac{\partial H}{\partial J_r} = -(-2h)^{3/2}\,.$$

According to the Bogoyavlenskij theorem it allows us to get bi-Hamiltonian formulations of these systems in the domain of definition of the action-angle variables (2.11,2.12).

2.2 Delaunay type variables

The action coordinates play an important role in classical dynamics because of their adiabatic invariance, i.e. invariance under infinitesimally slow perturbations. In the Kepler problem, there are few well-known types of orbits and, therefore, there are few types of the action-angle variables associated with different orbits. For instance, the Delaunay elements are valid only in the domain in phase space where there are the elliptic orbits [5]. On the other hand the two families of the Poincaré variables, which are the action-angle coordinates in the phase space of the Kepler problem, in the neighborhood of horizontal circular motions when eccentricities and inclinations are small [7]. There are also Delaunay-similar elements, Poincaré-similar elements and some other action-angle variables, which are well-defined in the neighborhood of different orbits.

For the perturbed Kepler problem we can also introduce the Delaunay type variables

$$\begin{aligned} &J_1 = J_\phi\,, &&J_2 = J_\phi + J_\theta\,, &&J_3 = J_r + \sqrt{\ell^2+\epsilon}\,, \\ &\omega_1 = \omega_\phi - \omega_\theta\,, &&\omega_2 = \omega_\theta - \frac{\ell}{\sqrt{\ell^2+\epsilon}}\,\omega_r\,, &&\omega_3 = \omega_r\,. \end{aligned} \tag{2.14}$$

Recall that the Delauney variables have a geometrical meaning directly related to the description of the orbits and their variations are much more significant for the astronomers than those of

Cartesian or spherical variables [7, 5, 17]. For $\epsilon = 0$ variables J_k, ω_k (2.14) coincide with the classical Delaunay elements $(\mathrm{l},\mathrm{g},\mathrm{h},\mathrm{L},\mathrm{G},\mathrm{H})$:

$$J_3 \equiv \mathrm{L} = \sqrt{a}\,, \qquad \omega_3 \equiv \mathrm{l} = n(t-\tau)\,,$$

$$J_2 \equiv \mathrm{G} = \sqrt{a(1-e^2)}\,, \qquad \omega_2 \equiv \mathrm{g} = \omega\,,$$

$$J_1 \equiv \mathrm{H} = \sqrt{a(1-e^2)}\,\cos i\,, \qquad \omega_1 \equiv \mathrm{h} = \Omega\,,$$

where n is the mean motion, a is the semimajor axis of the orbit, e is the eccentricity, i is the inclination, ω is the argument of the perigee, Ω is the longitude of the ascending node, τ is the time when the satellite passes through the perigee.

In the Delaunay type variables the Hamilton function H (2.13) takes the form

$$H = -\frac{1}{2J_3^2} \tag{2.15}$$

and, therefore, we can construct the bi-Hamiltonian formulation of the perturbed Kepler model with the second bivector P' given by (2.10). For instance, if

$$\beta_1(J_1) = J_1\,, \qquad \beta_2(J_2) = J_2\,, \qquad \text{and} \qquad K = -\frac{1}{3J_3^3}\,,$$

second bivector is equal to

$$P' = \sum_{k=1}^{3} J_k\,\frac{\partial}{\partial J_k} \wedge \frac{\partial}{\partial \omega_k} = \begin{pmatrix} 0 & 0 & 0 & J_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_3 \\ -J_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -J_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -J_3 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding recursion operator has three functionally independent eigenvalues which are the first integrals.

In the initial action-angle variables $(\omega_r, \omega_\theta, \omega_\phi, J_r, J_\theta, J_\phi)$ this bivector has a more complicated form

$$P' = \begin{pmatrix} 0 & 0 & 0 & J_r+\sqrt{(J_\phi+J_\theta)^2+\epsilon} & 0 & 0 \\ 0 & 0 & 0 & -\eta & J_\theta+J_\phi & 0 \\ 0 & 0 & 0 & -\eta & J_\theta & J_\phi \\ -J_r-\sqrt{(J_\phi+J_\theta)^2+\epsilon} & \eta & \eta & 0 & 0 & 0 \\ 0 & -J_\theta-J_\phi & -J_\theta & 0 & 0 & 0 \\ 0 & 0 & -J_\phi & 0 & 0 & 0 \end{pmatrix}$$

where

$$\eta = \frac{\ell(\ell - J_r - \sqrt{\ell^2+\epsilon})}{\sqrt{\ell^2+\epsilon}}\,, \qquad \ell = J_\theta + J_\phi\,.$$

In much the same way we can obtain other bi-Hamiltonian formulations associated with the two families of the Poincaré type action-angles variables or with other known types of the action-angle variables for the perturbed Kepler problem. Recall, for instance, that action variables at the first Poicaré type coordinate system are equal to

$$Z = J_\theta\,, \qquad \Gamma = J_r + \sqrt{\ell^2+\epsilon} - J_\phi - J_\theta\,, \qquad \Lambda = J_r + \sqrt{\ell^2+\epsilon}\,,$$

see [7] and references within.

Using similar action-angle variables for the relativistic Kepler problem we can also obtain the non degenerate bi-Hamiltonian formulation in contradiction with the Fernandes statement in [8].

3 Conclusion

Let us duplicate textually the Fernandes theorem from [8]:

Theorem: *A completely integrable Hamiltonian system is bi-Hamiltonian (satisfying (BH)) if and only if the graph of the Hamiltonian function is a hypersurface of translation,relative to the affine structure determined by the action variables.*

An additional assumption (BH) is that the corresponding recursion operator $N = P'P^{-1}$ has n functionally independent real eigenvalues $\lambda_1, \ldots, \lambda_n$.

This formulation of the theorem is often mentioned in modern literature, see for instance [1, 3, 4, 13, 14, 15, 16]. Nevertheless, there is some trivial misprint because the author omits one more necessary condition of the non degeneracy (2.7) of the Hamiltonian function, which could be found in the assumption "ND" on the page 5 in [8] and in the proof of the theorem.

The author forgets about this condition simultaneously in the formulation of the theorem and by considering physical examples of the applicability of this theorem. Thus, Fernandes [8] proclaims that perturbed Kepler problem does not have a bi-Hamiltonian formulation in contrast with the Kepler problem, see page 13 in [8]:

"Also we note that for the unperturbed Kepler problem ($\epsilon = 0$) the graph of the Hamiltonian is a surface of translation, and so it has a bi-Hamiltonian formulation (on the other hand, one can show that the relativistic Kepler problem also does not have a bi-Hamiltonian formulation)."

Let us repeat that initial and perturbed Kepler systems are degenerate systems and, therefore, we can not use the Fernandes theorem for both these systems simultaneously.

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References

[1] Błaszak M., Multi-Hamiltonian Theory of Dynamical Systems, Springer, 1998.

[2] Born M., *The Mechanics of the Atom*, G. Bells and Sons, Ltd, London, 1927.

[3] Bogoyavlenskij O.I., *Theory of tensor invariants of integrable Hamiltonian systems, I. Incompatible Poisson structures*, Comm. Math. Phys., 1996, v.180, pp.529-586.

[4] Boualem, H., Brouzet, R., Rakotondralambo, J. *About the separability of completely integrable quasi-bi-Hamiltonian systems with compact levels*, Differential Geometry and its Application, 2008, v. 26, n.6, pp. 583-591.

[5] Chang D.E., J. E. Marsden J.E., *Geometric derivation of the Delaunay variables and geometric phases*, Celestial Mechanics and Dynamical Astronomy, 2003, v.86(2), pp.185-208.

[6] Duistermaat J.J., *On global action-angle coordinates.* Comm. Pure Appl. Math., 1980 , v.33, no. 6, pp.687-706.

[7] Féjoz J., *On action-angle coordinates and the Poincaré coordinates*, Reg. Chaotic Dyn., 2013, v. 18(6), 6, pp. 703-718.

[8] Fernandes Rui L., *Completely integrable bi-Hamiltonian systems*, Journal of Dynamics and Differential Equations, 1994, v.6(1) , pp. 53-69.

[9] Magri F., *A simple model of the integrable Hamiltonian equation*, J. Math. Phys., 1978, v. 19, pp.1156-1162.

[10] Magri F., *Eight lectures on Integrable Systems*, Lecture Notes in Physics, Springer Verlag, Berlin-Heidelberg,1997, v.495, p.256-296.

[11] Marmo G., Vilasi G., *When do recursion operators generate new conservation laws?*, Phys. Lett. B., 1990, v. 277, pp. 137-140.

[12] Miranda E., *Integrable systems and group actions*, Central European Journal of Mathematics, v.12, pp. 240-270, 2014.

[13] Olver P.J., *Canonical forms for bihamiltonian systems*, In Integrable Systems (pp. 239-249), Birkhuser Boston, 1993.

[14] Sergyeyev A., *A simple way of making a Hamiltonian system into a bi-Hamiltonian one*, Acta Applicandae Mathematicae, 2004, v.83, n.1-2, pp. 183-197.

[15] Smirnov R.G., *Magri-Morosi-Gel'fand-Dorfman's bi-Hamiltonian constructions in the action-angle variables*, J. Math. Phys., 1997, v.38, n.12, pp.6444-6454.

[16] Smirnov R.G., *The action-angle coordinates revisited: Bi-Hamiltonian systems*, Rep. Math. Phys., 1999, v.44, pp.199-204.

[17] Vinti J.P., *Orbital and Celestial Mechanics*, AIAA, Reston, VA, 1998.